Problem Chosen	2022	Team Control Number
A	**MCM/ICM Summary Sheet**	**2217027**

The Secret to Win in Road Cycling Time Trial

Summary

Theoretically, athletic sport is an activity based on the characteristics as well as the potential of our body, for which a suitable and scientific theory can provide vital guidance to the athletes. In this paper, we are focusing on constructing the optimal models of the distribution of the power output. Both the consecutive condition and the discrete condition are considered in our work.

In the time trial course where the cyclists should ride against the clock, the best model can provide the shortest cost-time under certain limits. For such a optimal control problem, it is naturally to use Pontryagins principle to obtain the differential equation of the optimal function, while it is analytically solvable only when the race track is simplified.

However, the discrete model can solve the more complicated situations since the discretion can simplify the functional problem into a extreme value problem of a multivariable function.

The constraints of the optimal control problem come from the characteristics of the riders. Quantitatively, we determine the constraints as a power curve function and some functions of auxiliary rules according to the biological principles. Based on the optimal function and constraints rules, the optimal discrete control model t(P) can be determined.

Hence we apply the optimal model into the 2021 Tokyo Olympic and UCI World Championship Time Trial Course while considering the compare among the gender and all basic kinds of riders. It is noticeable that Time trial specialists take the lead in the above competitions, while the Sprinter is at a disadvantage. Analysing the data of the competitions, we summarize a golden law: Try to keep the speed constant, that means to accumulate the power output while the resistance increase. It is worthwhile to mention that too fast at the curve will lead to slide off the track and slow down while turning off is necessary. In addition, some potential factors such as the wind speed and multi-riders condition are considered to ensure the extensionality as well as sensitivity of our models. At the same time, we also find that the model has high fault tolerance and strong enforceability.

Keywords: Optimal control; Pontryagins principle; Euler-Lagrange Equation;Gradient Descent

Contents

1 Introduction

1.1 Background

Bicycle road race is one of the most important and significant sports events, especially the individual time trial part, in which the rider should try their best to reduce the riding time on a certain distance. As an approachable sport for lots of people, bicycle individual time trial has exerted great attention on its major global competitions such as The Olympic Time Trial course, World Championship time trial course, etc.

In the individual time trial course, the arrangement of the power output should be considered thoughtfully based on the individual specialty in order to shorten the riding time. The optimal strategy has been studied for a long time, authors in paper[1] propose a model considering cyclists fatigue dynamics, authors in paper[2] solved the optimal control problem by applying the Calculus of variations...these previous work quantitatively build up the optimal model and most of the them have a good correspondence from the experimental. Our work will go deeper based on these previous efforts.

1.2 Problem Restatement

We are supposed to determine a model suitable for all kinds of the cyclists. The model should consider the nature limitation of the cyclists as well as the potential influence from the environment(shape of the course, weather conditions, etc). Evaluation of the model is base on the application of the model as well as the sensitivity of the model.

As an optimal control problem, it is crucial to extract the major factors and design the distribution of the power output based on the factors. The dominant factors are derived from the analysis of the characteristic of both the environment and the cyclists. The best model can tell us the optimal strategy while inputting the major data, as long as the data is reasonable.

1.3 Our Work

Our work can be roughly divided into two parts: The construction of the optimal control models and the solution as well as the application of the models.

In the construction part, we build up both the optimal function as well as the constraints function in consecutive and discrete form, where the constraints mainly come from the power curve function.

In the solution and results part, we solve the optimal control problem and apply it to certain real courses.

Finally, we discuss some relevant factors about the extensionality and sensitivity of our models.

2 Construction of the Optimal Control Models: Let's play the Creator.

In this section, we will only focus on the model construction. As for the solution of the models, we will introduce it in the next section. After briefly pointing out the symbols and basic assumptions we use, we will gradually propose two models, namely, the continuous model and the discrete model. Considering that our model will gradually introduce a large number of parameters and gradually explain our "rules of the game" in many sub-sections, which may make the reader feel messy, we will make a clear summary at the end of this section and point out the constraints of each function and each parameter.

2.1 Assumptions

- When environment conditions are stable and constant, the best strategy of the riders is to stay at a certain level of the power output to balance the Resistance, while external output will cause a worse performance.
- The resisting force factor f reflect the conditions of the race track.
- Power curve is a characteristic of a rider.
- Exceeding the limits of the power curve need an extra compensation.

2.2 Notations

The primary notations are listed in the table2.2.

$Symbol$	$Definition$
CP	The highest mechanical power generated from aerobic energy
P_{max}	Limited maximal power
$P_m(t)$	Power curve
W'	The amount of mechanical work performed above CP
E	Total energy of a rider
E_i	The energy on specific race track
FT	Functional threshold
x	Distance from the origin
p(x)	Power distribution in the consecutive model
v(x)	Velocity distribution in the consecutive model
t(x)	Time spent on certain distance x
x_0	Origin of the specific race track
F	Drive Force
f	Resisting force
ξ	Resisting force factors
R	Radius of the Curve
$T_I(t_i)$	The accumulated time at t_{I_i} on a specific region I of the course

2.3 Construction of the Optimal Function and the Output

The riders in the time trial course are supposed to try their best to ran against the clock, which indicates that the time spend on a certain course is the Hamiltonian in the optimal control problem, that is *min* $\quad t = \int dt = \int \frac{dx}{v(x)}$.

Now we perform the force analysis on a rider shown in the figure(1,2,3), using the Newton's law we have :

$$N = mg \tag{1}$$

$$F - mg = ma \tag{2}$$

$$F^2 + (\frac{mv^2}{R})^2 = (\mu N)^2 \tag{3}$$

The distribution of the speed $v(x)$ directly depends on the desired power output $P(x)$of a rider.

$$P(x) = Fv(x) = (m\frac{dv(x)}{dt} + f)v(x) \tag{4}$$

where the f refers to the resistance.

Here comes the key point: the distribution of the power output $P(x)$ relies on both the state of the athlete and the least time motivation. In other words, we are dealing with a optimal control problem with state-constraint.

Since our task is to obtain the function P(x), it is naturally to apply the least action principle, which is not only elegant but powerful. Thus, we can simplify the problem of functional analysis into a differential equation, making the solution much more accessible.

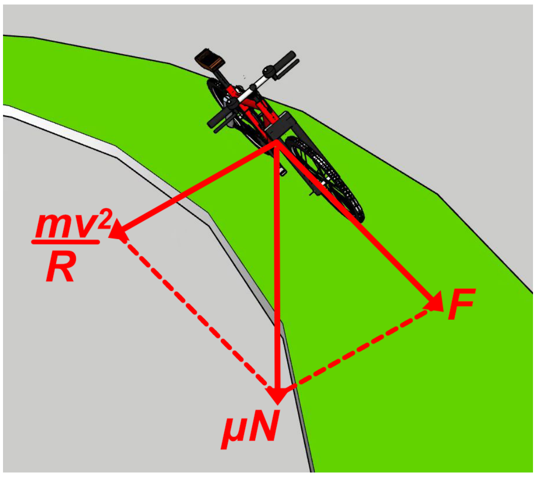

Figure 1: Force Analysis on the Curve

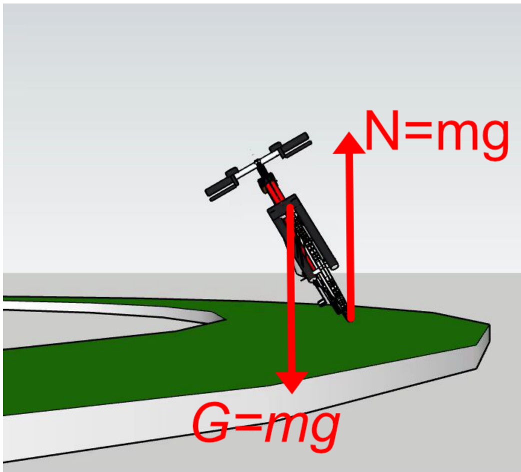

Figure 2: Vertically force analysis

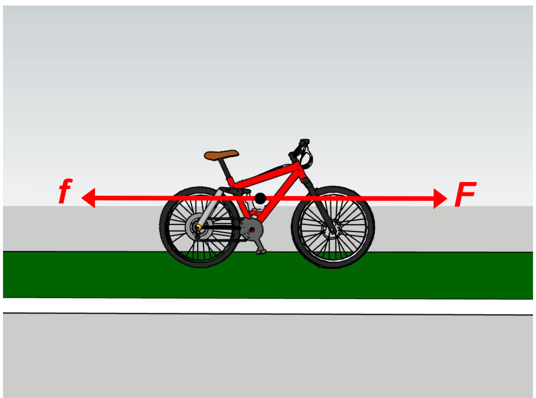

Figure 3: Horizontally force analysis

Figure 4: Cyclist on the race track

However, with the development of the complexity of the model, it is always hard to obtain the differential equation. If we did obtained the differential equation, the increasing complexity will possibly make it fail to satisfy the Lyapunov stability.

Therefore, it is worthwhile to consider the discretion strategy. According to the previous paper, the optimal strategy for a rider while the circumstance remains is to stay at a constant power output. In others words, an extra output of the power will lead to a worse condition when environment is constant. Thus, the long and complicated race track can be divided into several types of road based on some dominant indicators and we can define a resistance function of the race track model.

Hence in the discrete model, it is logically and reasonably to assume that the resistance function remain unchanged and uniform for each distance element.

Then we have the basic law in the discrete form as follows:

$$p_i t_i = F_i \Delta x \tag{5}$$

$$F_i = f + ma_i = f_i + m\frac{v_{i+1} - v_i}{t_i} \tag{6}$$

From the above two discrete equations, we can naturally get the recursive relations of t_is, and express them all by p_is and f_is. Finally, we sum all the t_is and get the function of the total time t with respect to p_is and f_is. Here p_is and f_is are still undetermined. We treat f_is as function parameters, p_is as an argument, or more accurately, the expected p_is allocation scheme as an argument. Finally, the solution process of our optimization algorithm is to get the optimal allocation scheme of p_is, so that the total time t can be minimized. The specific functional relationship will be determined later.

2.4 Characteristics of the Riders

As we all know, the basic characteristics of a cyclist should reflect their ability of delivering rapid bursts of energy as well their endurance, such as the neuromuscular power, anaerobic capacity, maximal oxygen uptake and lactate threshold, etc.

Based on the previous work from the some experts, we extract some dominant parameters used to decribed a rider, namely: W', P_{max}, CP and FT. According to Michael J. Puchowic et al, W' is the amount of mechanical work that can be porformed above CP, and CP, also known as the metabolic threshold, represents the highest mechanical power that is wholly generated from aerobic energy. Any power exceeding CP draws upon W', the depletion of which leads to fatigue and task failure. P_{max} represent the maximal power the cyclist can generate and FT, which is short for Functional Threshold, is a critical time limit beyond which the cyclist might exhaust and the performance of aerobic exercise may be affected. The maximum power of cyclists will be limited to CP once his or her W' is exhausted.

However, the above parameters are not enough to depict a cyclist while riding. A power curve indicating how long one can generate a certain amount of power is also important in the description for a rider, while the real output P above the Pm(T_I) will lead to a long recover time.According to the model we mentioned later, the above parameters will appear as important parameters of the power curve, and we also need some features to describe the rider's ability to briefly exceed the limit of the power curve.

Thus, we propose a feature called Sprint (SP, for short) to depict the ability of a rider to reach a higher level (concretely, $P_i > P_m$), of which the unit is J/kg. Correspondingly, a feature called Energy Accumulation (EA,$J/kg \cdot s$)is introduced to describe the consumption of SP.

Figuratively speaking, the SP is like the accumulator bar in a racing game. The energy of the accumulator bar SP can be used to support the part of the rider's power that exceeds the limit. Only when the accumulator bar SP is full can the rider make such a sprint. When the rider does not sprint, the accumulator bar recovers at the rate of EA. This is just a visual statement, and the numerical relationship will be sorted out in the summary of this section.

Using the six dominant features of a rider, the radar ability distribution of four usual types of cyclist is shown in figure(5), where the parameters is extracted from the power curve model we build up in the next subsection.

2.5 The Power Curve Model [Problem 1]

The power curve of a certain cyclist is a roughly quantitative description of his or her physical fitness. The curve is plotted under the time and power axes and is traced out by the maximal power the cyclist can maintain during given amount of time.

The power curve can be plotted using experimental measurement, and there are various

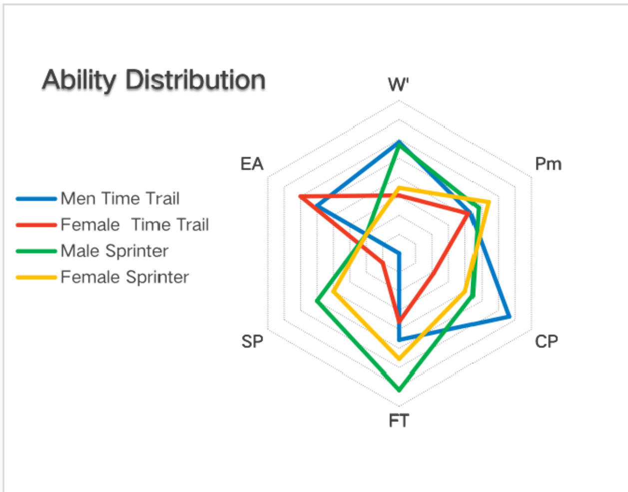

Figure 5: Result of curve fitting, parameters for male and female time trialists and sprinters

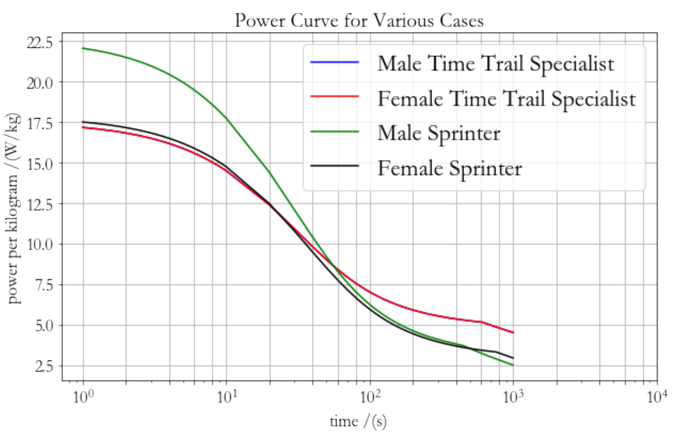

Figure 6: Result of curve fitting, parameters for male and female time trialists and sprinters

models that can fit and describe the behavior of the curve, including CP model (Monod & Scherrer, 1965), exponential model (Hopkins, Edmond, Hamilton, Macfarlane, & Ross, 1989), 3CP model [3] and the developed OmPD model[4], which is the model that's used in this article.

The formulation of OmPD model[4] is the following.

$$\begin{cases} P(t) = \frac{W'}{t}\left(1 - e^{-\frac{P_{max}\ -\ CP}{W'}t}\right) + CP, & t \leq FT. \\ P(t) = \frac{W'}{t}\left(1 - e^{-\frac{P_{max}\ -\ CP}{W'}t}\right) + CP - A \cdot \ln\left(\frac{t}{FT}\right)\ , & t > FT. \end{cases} \tag{7}$$

Data that can be accessed on the website (https://www.trainingpeaks.com/blog/power-profiling/) are used for further determination of the parameters of different classes of cyclists. The data of male and female Time Trialists and Sprinters are used for curve fitting to determine the four parameters for male and female time trialists and sprinters.

The result of curve fitting is displayed in Figure(6). Based on the Ompd model and the actual experimental data, we can fit the data to determine the parameters of different types of cyclists shown in the figure(5). They show the power curve and OmPD parameters for male and female time trail specialists and sprinters.

2.6 The Race Track Model [Problem 2]

The race track in the time trial competition is complicated due to the abundant types of the geomorphy on earth. Therefore, we extract the dominant factors of the race track that can exert the most important influence on the athletes. In other words, the influence of the types of the race track can be described by a few dominant parameters of the road. Now we define the total influence as the resistance f, which is a function of the given parameters.

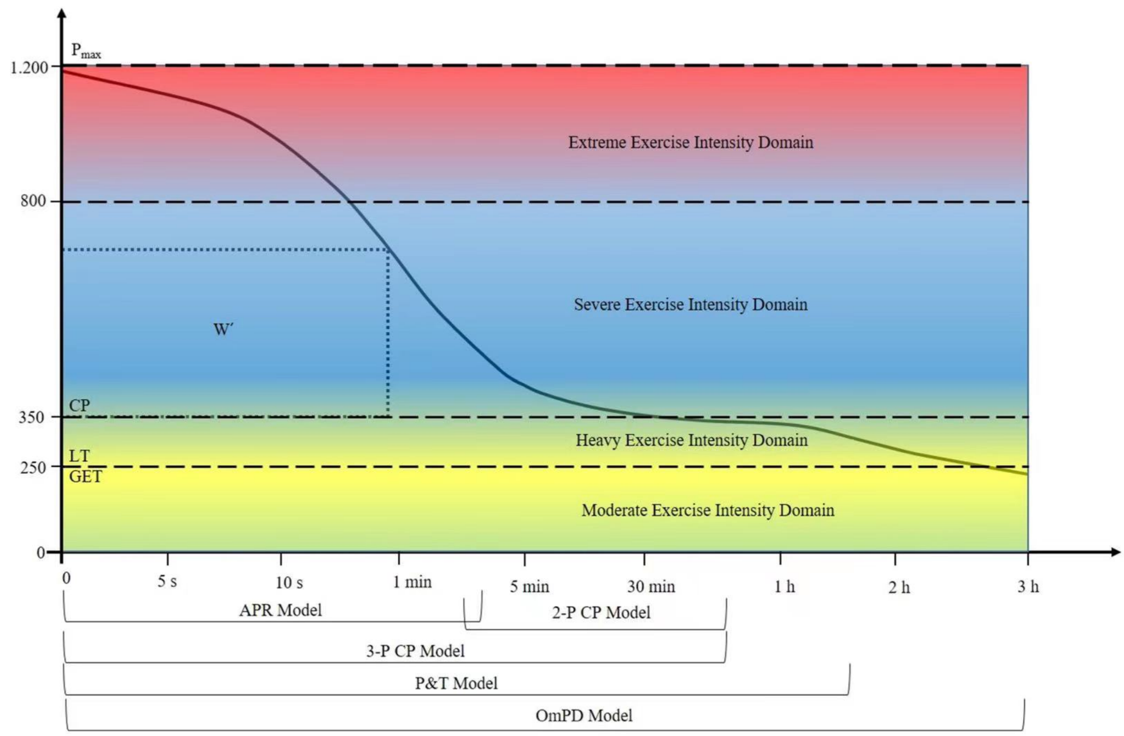

Figure 7: Comparison between various models, and OmPD Model fits the best [4]

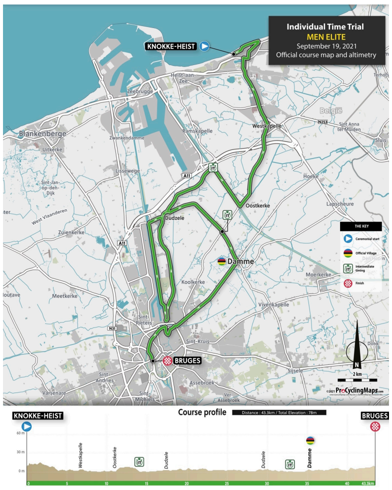

Figure 8: Map of the UCI world Championship Time trial course[5]

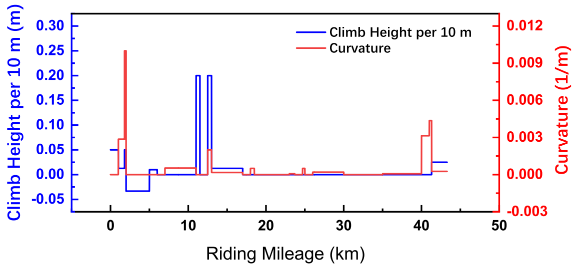

Figure 9: Height and Radius Distribution of Curve of the UCI Time trial Course

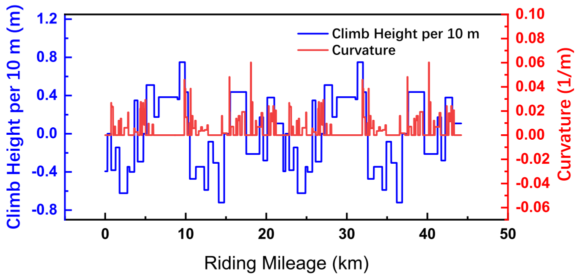

Figure 10: Height and Radius Distribution of Curve in the Tokyo Olympic Time trial Male Course

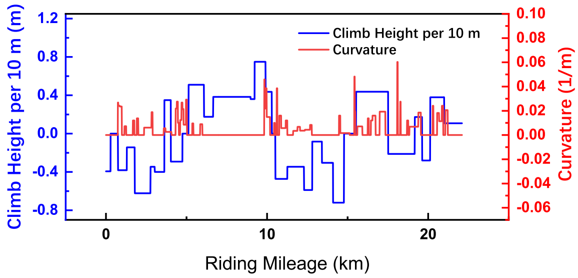

Figure 11: Height and Radius Distribution of Curve in the Tokyo Olympic Time trial Female Course

Theorectically, the gradient and the curvature of the race track is the major factors in the time trial course. It is worthwhile to mention that the speed of the wind on a specific road also topologically rely on the shape of the road itself. Thus we make a bold but reasonable assumption that the influence of the wind can be half quantitatively including in the race track model.

In conclusion, our race track model take the gradient η, speed of wind ω, the curvature of the road ρ into consideration.

Therefore, the coarse model of the resistance is as follows.

$$f = mg\eta + \xi S(v + \omega) \tag{8}$$

where F is the resistance, v is the speed of the rider, S is the cross-sectional area of the rider, ξ is the resistance coefficient .

while the discrete form is as follows,

$$f_i = mg\frac{\Delta h_i}{\Delta x_i} + \xi S(v_i + w_i) \tag{9}$$

where Δh_i is the relative height on a specific distance element identified by i, v_i is the speed of the athlete.

The influence of the curvature can be derived by the force analysis shown in figure1, from which we obtained the following inequality constrain:

$$\frac{v^4}{R^2} + f^2 \leq (\mu m g)^2 \tag{10}$$

where $R = \frac{1}{\rho}$. Above all, our race track model use three key features (Curvature , gradient, speed of the wind) to describe a real road.

In order to build up the race track model numerically, we extract the distribution of the three types parameters from the course data of 2021 UCI World Championship Time Trial Course(http://www.steephill.tv/road-cycling-world-championships/) as well as the Tokyo Olympic Time Trial Course. And the distribution of the height and the curvature are shown in figure(8,9,22).

2.7 Summation of the Model Construction

The relations above can be vividly described in the figure(12).

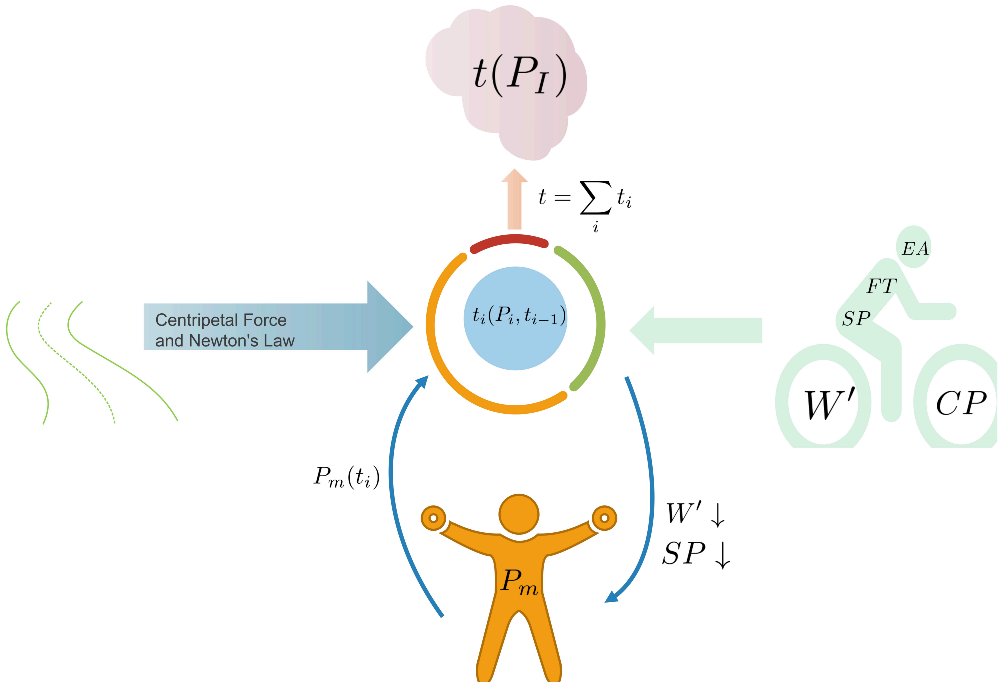

Figure 12: Summation Relation

To sum up, the **Consecutive Optimal control model** can be restated as follows:

$$min \quad t = \int \frac{dx}{v(x)} \tag{11}$$

with the following constraints:

$$\frac{dx}{dt} = v(x) \quad x \in [0, x_f] \tag{12}$$

$$\int P(x)dt = \int \frac{P(x)}{v(x)}dx = E \tag{13}$$

$$P(x) = (m\frac{dv(x)}{dt} + f)v(x) \tag{14}$$

as well as the boundary condition:

$$v(0) = 0 \tag{15}$$

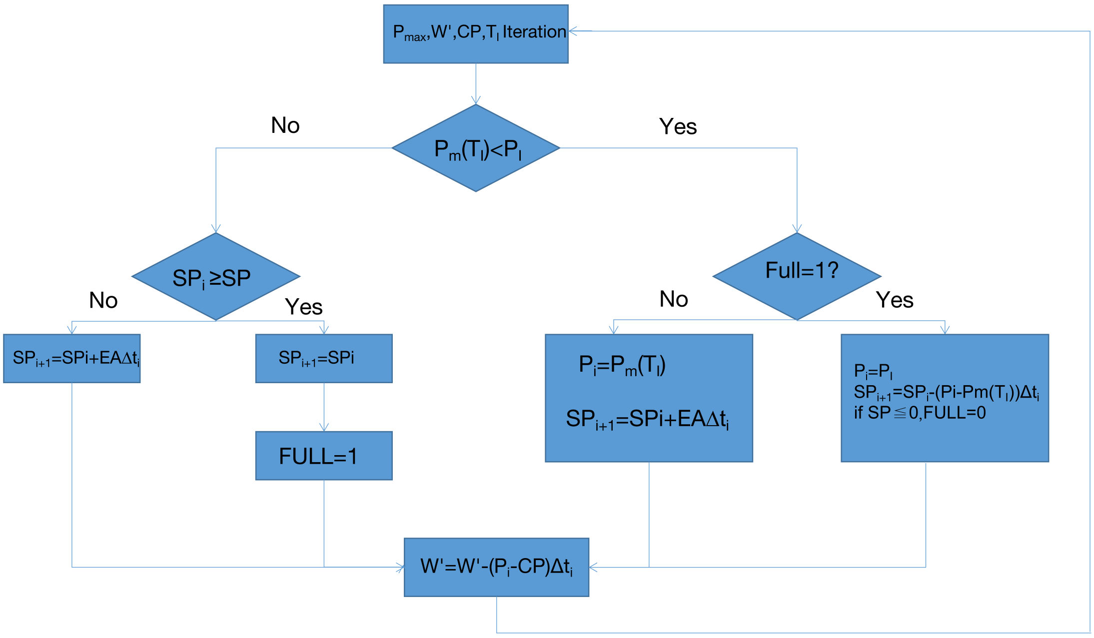

Figure 13: Restriction Relationship Between Power Curve and Rider's Actual Power

The **Discrete Optimal Control Model** can be restated as follows:

$$t = \sum_i t_i \tag{16}$$

In the next section, we consider the combinations of various types of race track, and divide the whole race track into K parts, then the equation is rewritten as follows:

$$t = \sum_{I=1}^{K} \sum_i t_{I_i} \tag{17}$$

$$p_i t_i = F_i \Delta x \tag{18}$$

where the drive force F can be written as the following form based on the race track model.

$$
\begin{aligned}
F_i &= eq\eta S v_i + (\eta S \omega_i + mg\frac{h_i}{\sqrt{\Delta x^2 - h_i^2}}) + ma_i \\
&\equiv \alpha v_i + \beta_i + \frac{v_{i+1} - v_i}{t_i} \\
&\equiv \frac{\alpha \Delta x}{t_i} + \beta_i + \frac{m\Delta x}{t-i}(\frac{1}{t_{i+1}} - \frac{1}{t_i})
\end{aligned}
$$

Thus we obtain the time we cost at a certain distance element i:

$$
t_{i+1}(p_i) = \frac{m\Delta x^2}{p_i t_i^2 + \frac{m\Delta x^2}{t_i} - \alpha \Delta x^2 - \beta_i \Delta x t_i} \tag{19}
$$

The above formula shows how to use p_i to calculate $t_i$and p_i is obtained through the calculation flow shown in figure(2.7). In constant iterations, we calculate pi and ti. First of all, we judge that the expected power P_I in the I^{th} road we allocate exceeds the $P_m(T_I)$ given by the power curve, where T_I shows the time the rider has spent on the I^{th} road. If it exceeds the limit, we have to sprint according to whether the rider's SP is full or not, or force to reduce the power to meet the limit. In the non-sprint state, the SP will be supplemented, such as equation(20). The consumption of SP in the sprint phase is equation(21). At the same time, if the rider's real-time power exceeds his characteristic CP, its anaerobic exercise capacity W' will be reduced in the form of equation(22), which will bring negative gain to his power curve.

$$
SP_{i+1} = SP_i + EA\Delta t_i \tag{20}
$$

$$
SP_{i+1} = SP_i - (p_i - P_m(T_I))\Delta t_i \tag{21}
$$

$$
W'_{i+1} = W_i - (p_i - CP)\Delta t_i \tag{22}
$$

At this point, we have fully determined how to get the total time from our expected power allocation, and the next task is to optimize our expected power allocation with the goal of minimizing the total time.

Above all, we build up two optimal control model, the consecutive one and the discrete one, while their advantages and disadvantages will be shown entirely in the next section.

3 Solution and Results of the Optimal Control Models: Model solving: Let's play time trial athletes!

3.1 Solution and Results of the Continuity Model

In the time trial course, riders should spare no effort to ride against the clock. In this part, our task is to solve the optimal control with the state constrains based on the Pontryagins principle. The optimal function $p(x)$ can be determined by the optimal distribution of $v(x)$ according to the equation(14), thus we could focus on figure out the optimal function $v(x)$. Now we restate the problem :

$$
min \quad t = \int_0^s \frac{dx}{v(x)} \tag{23}
$$

The boundary condition is

$$v(0) = 0, v(x_f) > 0 \tag{24}$$

Obviously, it is a half free boundary condition. Considering the race track is a simple type and the Resistance f is of the simplest form:

$$f = \xi v(x) \tag{25}$$

The integration state constraint can be derived by the chain rule :

$$\int mv(x)\frac{dv(x)}{dx} + \xi v(x)dx = E \tag{26}$$

According to the work of J. Frederic Bonnans[6] , the problem is actually a optimal control of state constrained integral equations, of which the Hamiltonian can be written as follows according to the Pontryagins principle :

$$H = \frac{1}{v(x)} + \lambda(x) \cdot [mv(x)\frac{dv(x)}{dx} + \xi v(x)] \tag{27}$$

where the $\lambda(x)$ is the lagrange multiplier. Though with the Euler-lagrange Equation, it is still difficult to obtain the concrete form of $v(x)$ due to the unknown multiplier.

Therefore, some simplification strategies are used to obtain the optimal function. Firstly, according to the assumption 1 and thus numerically, $v'(x) << 1$. Secondly, we use the final speed as the normalization factor for a given total energy. Hence, we can obtain the differential equation constraints as follows:

$$v^2(x)v'(x)^2 + v(x)v'(x) - v'(x) = 0 \tag{28}$$

Thus the revised functional problem is restated as follows:

$$t^* = \int \frac{1}{v(x)} + \lambda[v^2(x)v'(x)^2 + v(x)v'(x) - v'(x)]dx \tag{29}$$

Thus we can obtain the differential equation using the Euler-Lagrange equation

$$\lambda v'(x)^2 \cdot v(x)^2 - \frac{1}{v(x)} + \lambda v(x) = 0 \tag{30}$$

The result of the differential equation is as follows:

3.2 Solution and Results of the Discrete Model

Guided by the discrete optimal control model discussed in previous section, an algorithm is designed, which takes in the power distribution during the whole tournament, the type of the cyclist under discussion and possible road conditions and returns the expected time of the race. Then, certain optimization algorithms are used to figure out the suggested power distribution. We have tried multiple algorithms including **Nelder-Mead, Powell, GC** and **Sequential Least Squares Programming**, and it is Sequential Least Squares Programming (SLSQP) that actually worked under this paticular circumstances.

Male and female time trail specilists and sprinters are analysed and the results are displayed from figure.14 to figure.21.

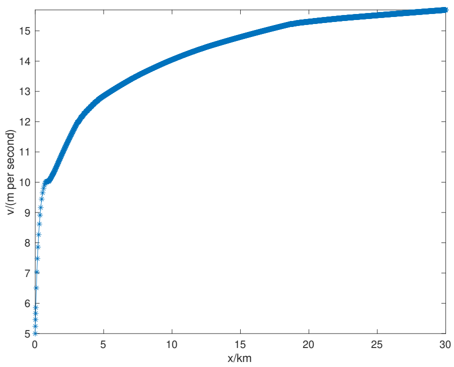

Figure 14: Speed distribution

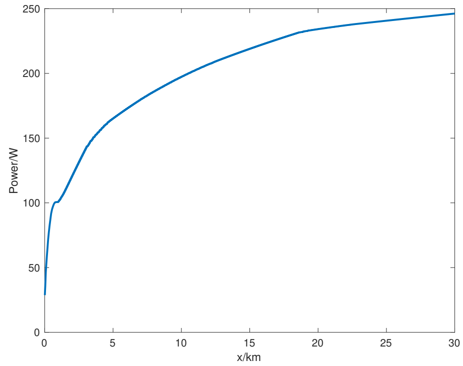

Figure 15: Distribution of the Power Output

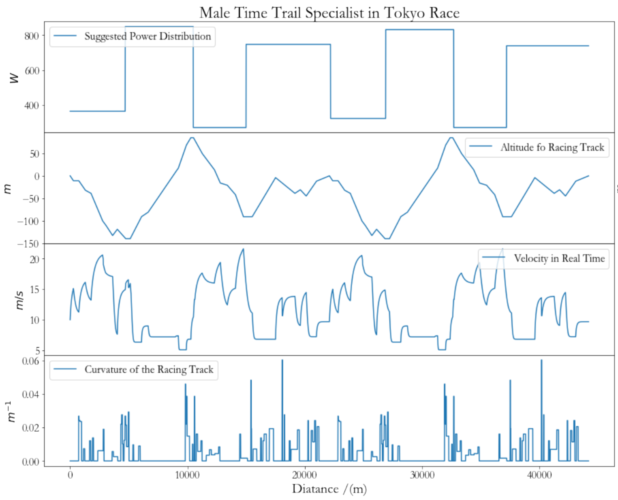

Figure 16: Simulatoin result of male time trail specialist in Tokyo Olympic Time trial Male Course in detail

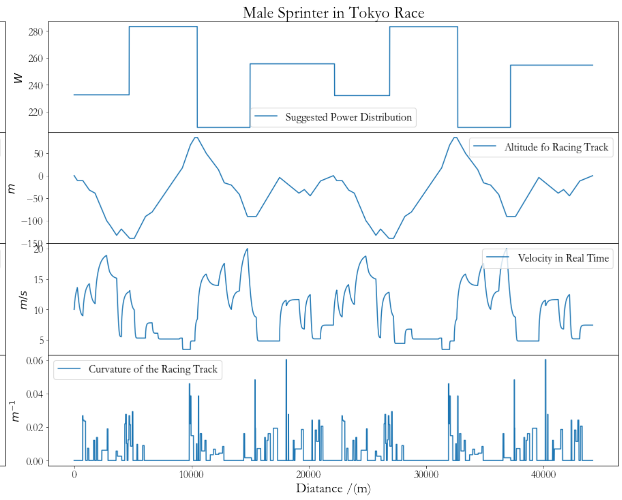

Figure 17: Simulatoin result of male sprinter in Tokyo Olympic Time trial Male Course in detail

Figure 18: Simulatoin result of female time trail specialist in Tokyo Olympic Time trial Female Course in detail

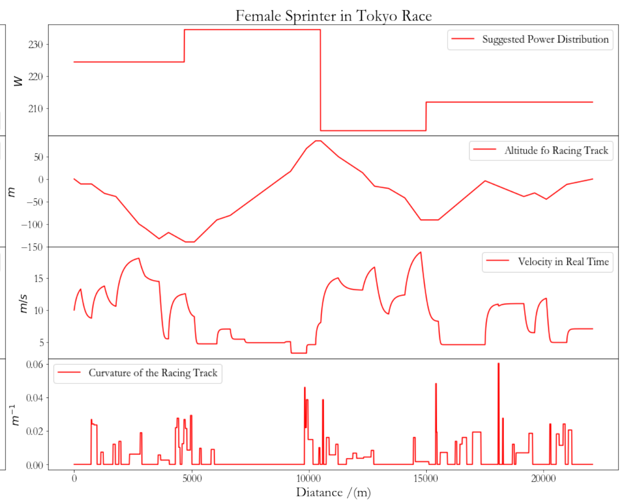

Figure 19: Simulatoin result of female time trail specialist in Tokyo Olympic Time trial Female Course in detail

3.2.1 Who will be the winner of the 2021 Tokyo Olympic Time Trial Course? [Problem 2]

The result of the simulation are listed in Table.1, from which one can tell that time trail specialists will be the winner in both genders.

3.2.2 Who will be the winner of the 2021 UCI World Championship Time Trial Course? [Problem 2]

The result of the simulation are listed in Table.2, from which one can tell that time trail specialists will be the winner in both genders.

4 Discussion

This section will first be focusing on the interpretation of the model described and simulated in previous sections. Then, influence of weather conditions, the speed of wind in paticular, and curvature of the track are taken into consideration, after which follows the sensitivity analysis and the team time trail model.

4.1 Potential Influence: God Bless Us! [Problem 2&3]

A customized race track is constructed in this section, which contains not only the altitude, but the curvature and the strength of the wind as well. MCM Road Cycling track is specially designed to commemorate the 2022 MCM Competition. As shown in figure(24), the MCM track is characterized by a long straight and many sharp turns. There's a long uphill, at the letter

Gender	Type	Expected time/second	result
Male	Time Trail Specialist	3967.95	Winner
	Sprinter	4999.25	Loser
Female	Time Trail Specialist	4224.35	Winner
	Sprinter	5254.91	Loser

Table 1: Expected result of Tokyo race

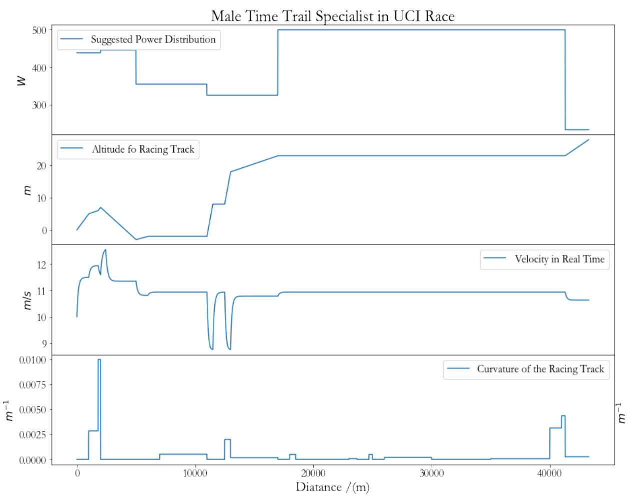

Figure 20: Simulatoin result of male time trail specialist in 2021 UCI World Championship Time Trial Course in detail

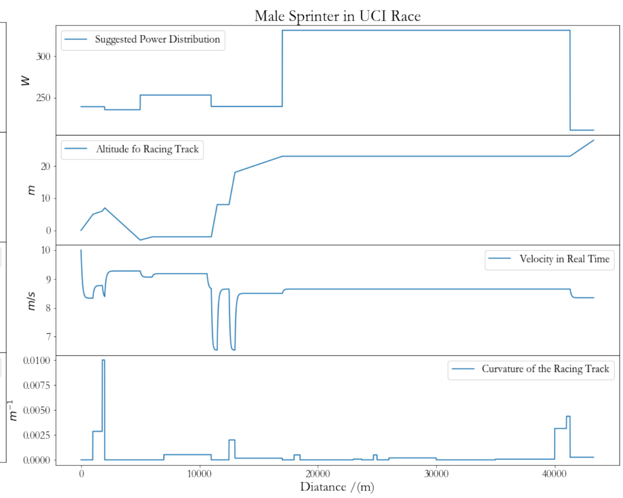

Figure 21: Simulatoin result of male sprinter in 2021 UCI World Championship Time Trial Course in detail

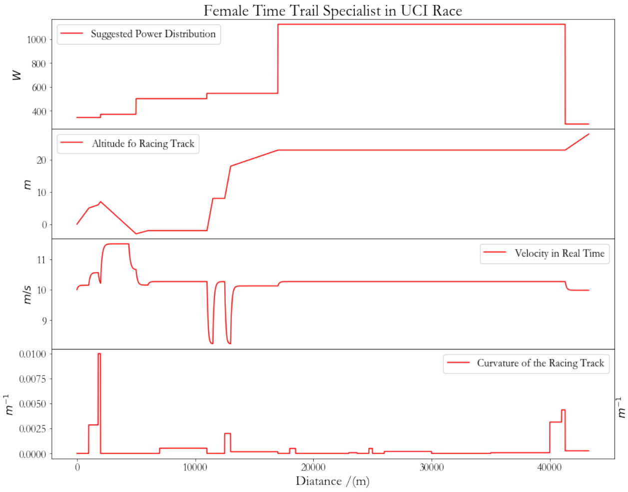

Figure 22: Simulatoin result of female time trail specialist in 2021 UCI World Championship Time Trial Course in detail

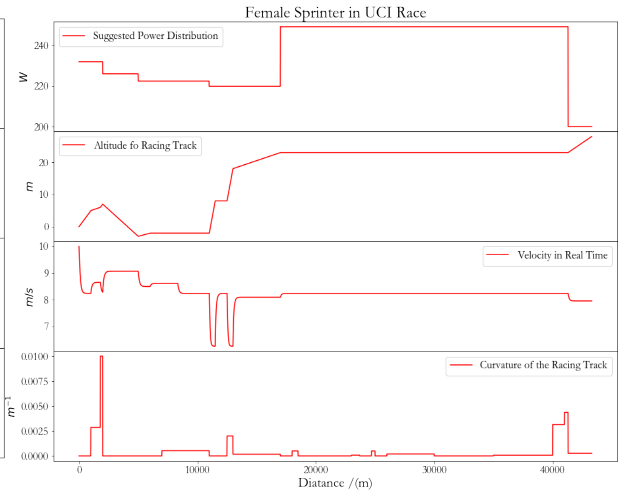

Figure 23: Simulatoin result of female sprinter in 2021 UCI World Championship Time Trial Course in detail

Gender	Type	Expected time/second	result
Male	Time Trail Specialist	4347.11	Winner
	Sprinter	5664.80	Loser
Female	Time Trail Specialist	2295.20	Winner
	Sprinter	2941.33	Loser

Table 2: Expected result of 2021 UCI race

"C". Its weather environment is also unique, the wind is blowing from all around to the center, and there will be no wind, breeze, strong wind three kinds of wind. The wind speed component data of the road direction have been shown in figure(25, 26, 27). The Strength of the wind is adjusted to manifest the influence of the wind.

Male time trail specialist model is used through out the simulations, which is displayed in figure(25, 26, 27).

From the result of the simulation, one can see that generally for a time trail specialist, the model suggests that he saved power while climbing up hills and ride faster when the altitude drops. the role that curvature of the track played is a potential brake, during which the cyclist has to slow down to avoid drifting off the track. The strength of wind will affect the speed of the cyclist, but it dose not seem to affect the suggested plan for the rider. This phenomena can be understood by the fact that the air resistance, which is set to be proportional to the relative velocity of man and moving air in accordance with the fact that the velocity is less than the Reynolds number of the drag form conversion, is relatively small compared with the drag of gravity when climbing up hills.

4.2 Sensitivity Analysis: So Strict? So Easy! [Problem 4]

In fact, our power allocation scheme itself has a fairly good fault tolerance, because our method of dividing the road into several sections and allocating constant power to each section is very enforceable. In order to verify this point, we disturb the optimal scheme of the model to a certain extent and observe the time-consuming increment caused by it.

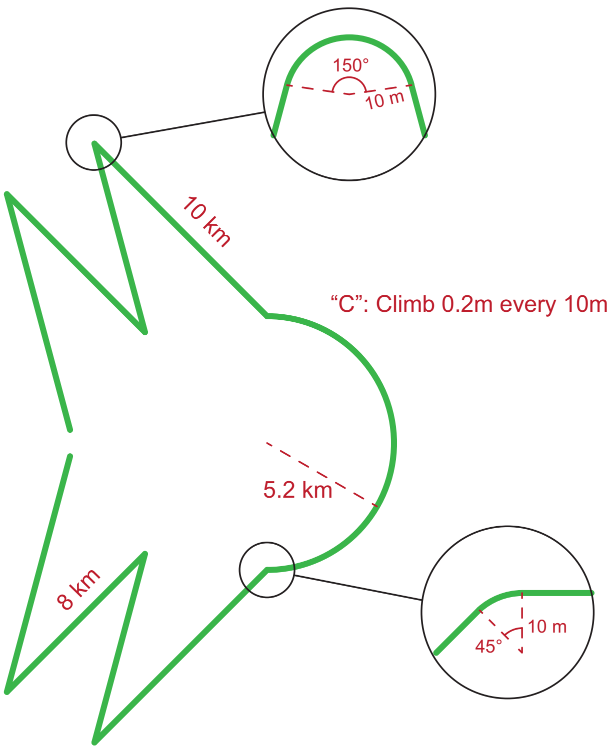

Figure 24: "MCM" Time Trial Course

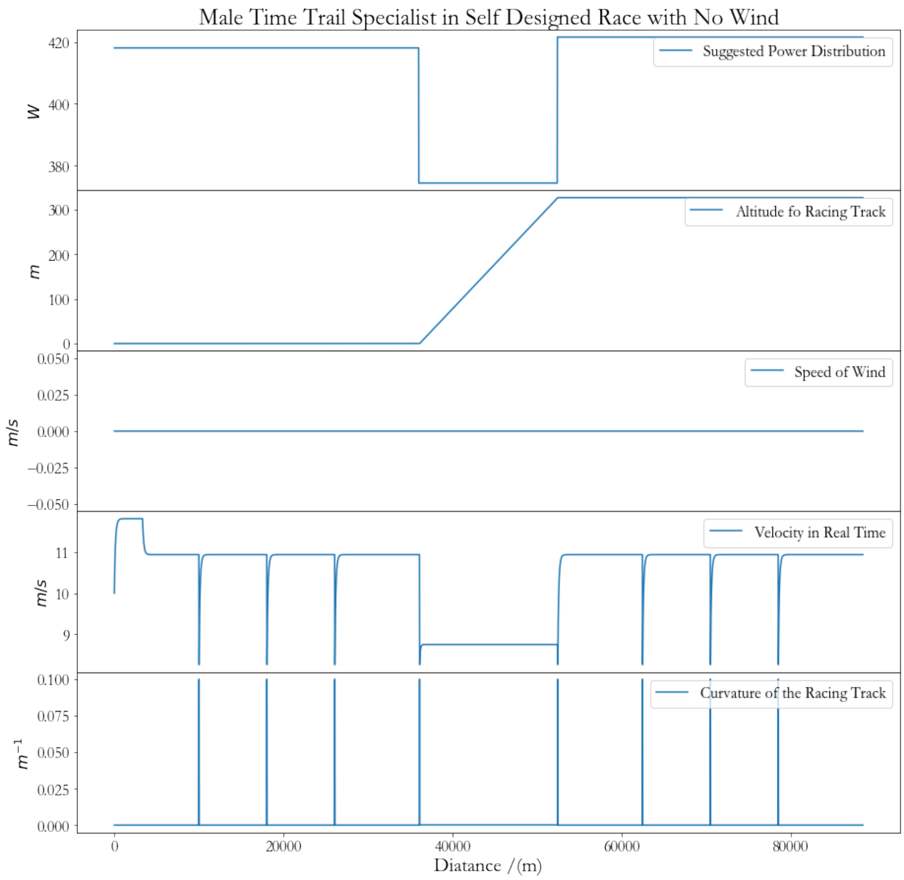

Figure 25: Simulation result of male time trail specialist in customized race track with **NO Wind**

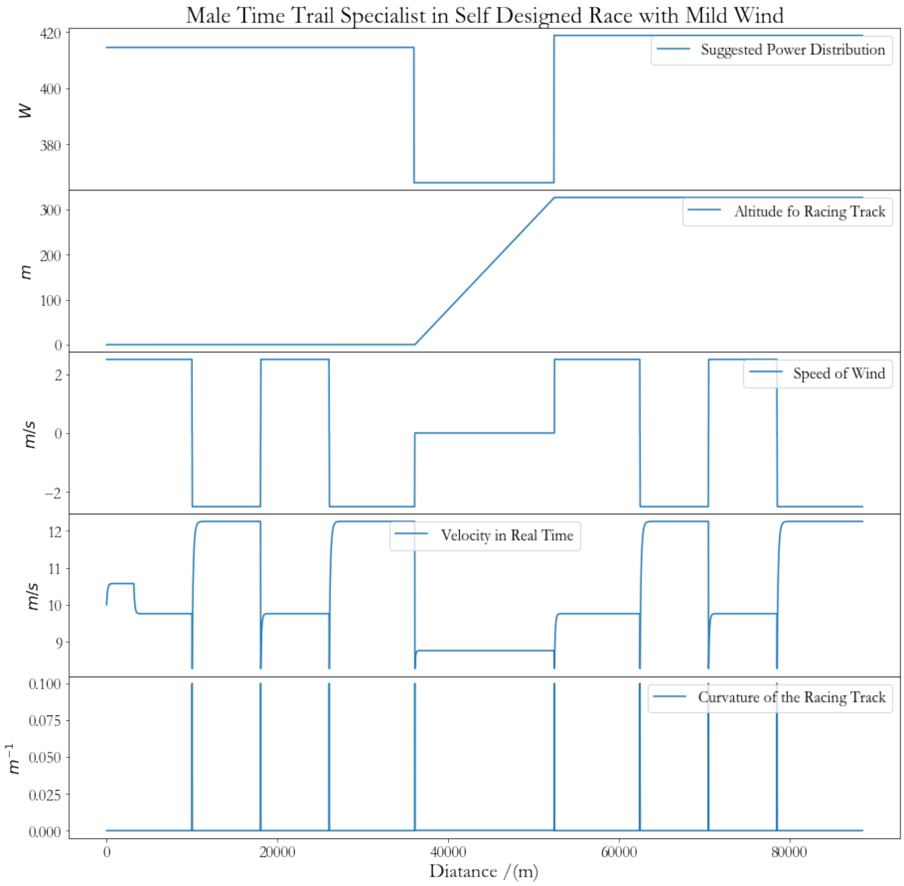

Figure 26: Simulation result of male time trail specialist in customized race track with **MILD Wind**

We note that the optimal expected power allocation scheme is

$$\mathbf{P_0} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{bmatrix} \tag{31}$$

We disturb p to a certain extent

$$\tilde{\mathbf{P}} = \begin{bmatrix} P_1 + \Delta P_1 \\ P_2 + \Delta P_2 \\ \vdots \\ P_N + \Delta P_N \end{bmatrix} \tag{32}$$

And the disturbance rate is defined as a ratio of second order norm

$$\mathrm{Tol} = \frac{\left\|\tilde{\mathbf{P}} - \mathbf{P_0}\right\|}{\|\mathbf{P_0}\|} \tag{33}$$

We substitute the power allocation after disturbance into the model to calculate the time-consuming increment, which is shown in figure(28,29). And according to the response of the average time-consuming increment to the power tolerance, we get: There is a tolerance of 8.35% between the athlete's power allocation and the strict scheme with an error of 2 seconds of his total time.

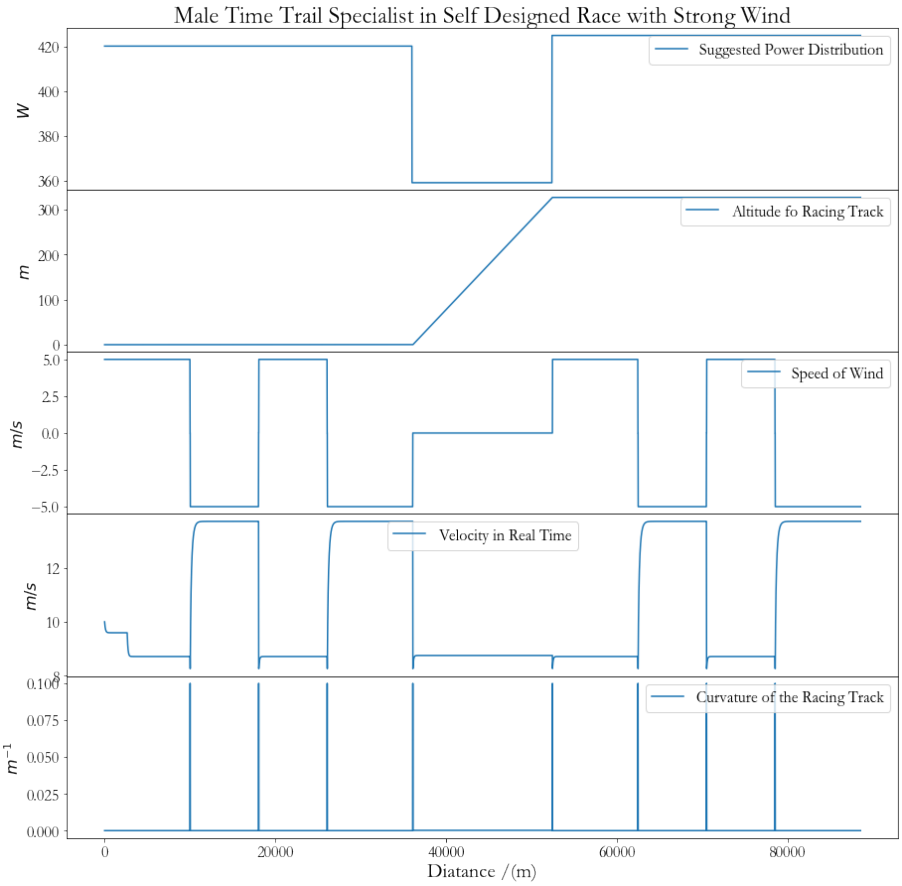

Figure 27: Simulation result of male time trail specialist in customized race track with **STRONG Wind**

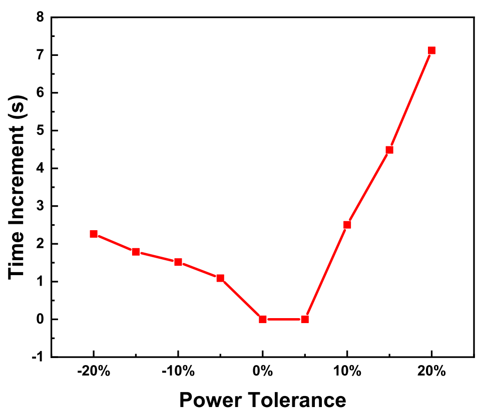

Figure 28: Variation of Time-consuming Increment with Power Tolerance

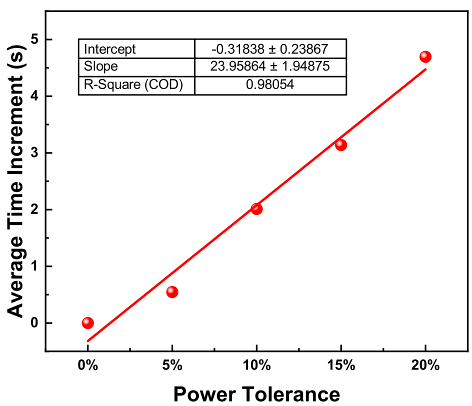

Figure 29: Average Variation of Time-consuming Increment with Power Tolerance

4.3 Team Work: Winner Winner, Chicken Dinner! [Problem 5]

Our model has good portability, which brings unlimited potential to our model expansion.

Now we can consider optimal power use for a team time trial of six riders per team, where the teams time is determined when the fourth rider crosses the finish line. We marked six people on a team as

$$q = 0, 1, 2, 3, 4, 5 \tag{34}$$

Then the function of calculating the total time in our personal time trial model can be directly by marking extended to

$$t(P_I) \longrightarrow t^{(q)}(P_I^{(q)}) \tag{35}$$

where $P_I^{(q)}$ represents the expected power distribution of individual q in road I.

However, the biggest strategic difference between team time trial and individual time trial is to consider the cooperation between team members. In general, following team members can reduce a large part of the wind resistance, so we need to consider the distribution of power among different members. This can be simply and effectively reflected in the coupling between $P_I^{(q)}$, and we consider such a coupling mode: For each Q, determine whether the distance difference between the other members and it (more directly, the difference in the index of the distance microelement) is less than a given critical value, that is,

$$\text{if} \quad i^{(other)} - i^{(q)} <= i_{cr} \tag{36}$$

If such a wind shield condition is satisfied, we will multiply the term before the wind resistance in the model by a smaller coefficient

$$f = \gamma\eta S(v_i + \omega_i), \quad (\gamma < 1) \tag{37}$$

In this way, we get the coupled member time-consuming functions, and then we sort them and re-mark q from less to more time.

$$t_{sort}^{(q)} = t_{sort}^{(q)}(P_I^{(0)}, P_I^{(1)}, \cdots, P_I^{(5)}) \tag{38}$$

Finally, we set the goal of minimization as

$$\text{minimize} \quad t_{sort}^{(q)} \tag{39}$$

The remaining work is the same as the solution of the previous model, which is still a simple problem of finding the extreme value of multivariate function, so we won't repeat it any more.

5 Validating the Model–Strenghs and Weaknesses

5.1 Strengths

- The construction of the model is reasonable and physically.
- The diversity of the optimal model while considering the consecutive model in the simple mode and the discrete model in the complicated mode.
- The accuracy of the solution of the models are acceptable and reasonable.

5.2 Weakness

- The assumptions in the simplified consecutive model should be pondered.
- The discrete model inevitably simplifies the road conditions, which makes the prediction of competition results may not be accurate enough.
- Many parameters are introduced into the model, but there is a lack of sufficient experimental data to accurately determine these parameters.

6 Conclusions

In this paper, we construct both the consecutive model as well as the discrete model of the optimal control problem, while the former used to solve the simple type of race track and the latter is widely used in the complicated course. The constraints of the optimal problem come from the power curve and some auxiliary biology rules, all of which is extracted either from the fit of the real data or from previous work.

We also apply our model on various types of courses based on our race track model considering both the extensionality and the sensitivity of our models.

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Appendices

Appendix A Rider's Race Guidance for MCM Time Trail Course

Entrusted by numerous riders and Directeur Sporif, This appendix includes a rough guidance for the rider and Directeur Sporif, which is based on the model that we developed in previous context and is suitable for the case of the single rider, one time trail course. The target cyclist type for this guide are time trail specialists and sprinters.

A.1 Introduction to the MCM Time Trail Course

The MCM time trail course is a 88550 meter-long course with slight climb in the middle. As shown in figure(30), the MCM track is characterized by a long straight and many sharp turns. There's a long uphill, at the letter "C". Its weather environment is also unique, the wind is blowing from all around to the center.

A.2 Typical Information of Cyclists

The information, characterized by the parameters: W', P_{max}, CP, FT, SP and EA for typical male and female cyclists is displayed in figure.31. The meaning of those parameters will be omitted since riders and Directeur Sporif are familiar with the meanings.

A.3 A Casual Walk Towards The Model

The model puts the altitude, the degree of curve of the course the strength of the wind and parameters of certain cyclist into consideration. And the aim of the model is to analysis the information given above to produce a guidance on how should the cyclist distribute power on different parts of the track.

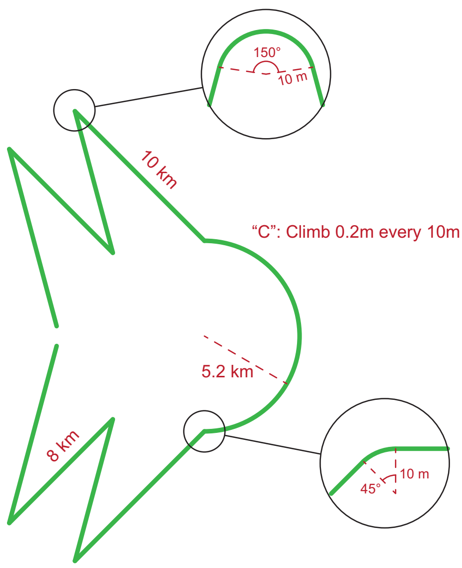

Figure 30: Top view of the MCM time trail course

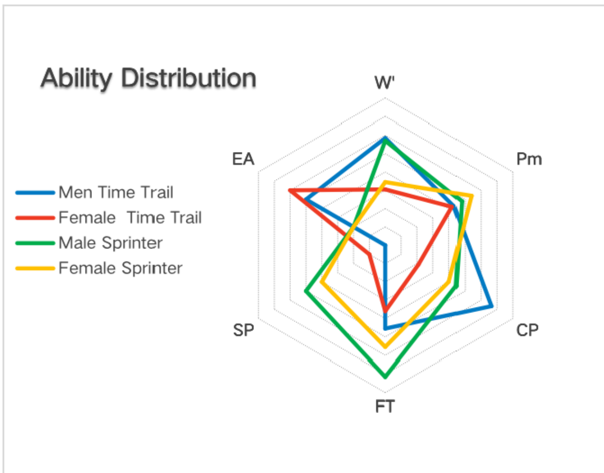

Figure 31: ing, parameters for male and female time trialists

The model is based on rigorous analysis of the physical and dynamical process, and our model simulates the real situation of the game as truly as possible through these parameters, and tries to find the most optimal power allocation scheme with many trial and error. Further more, the optimization is based on various algorithms, which makes our model more convincing.

A.4 Strategy to Win The Course

From the analysis of the model, we provides the following suggestions on how a typical cyclist should distribute his or her power:

A.4.1 General Rule

- The cyclist is suggested to maintain the speed at a level which should be neither too fast nor too slow.
- When climbing, the cyclist should increase power while harness the power of downhill to save energy.
- Ride harder when the wind is headed, and use the wind to save energy when the wind is down.
- Control your speed around the turns to avoid drifting off the track, and try to make as wide a circle as possible, this will allow you to have higher speed when turning

A.4.2 Additional Rules for Time Trail Specialists

- Time trail specialists generally have higher endurance, so in this case, the increase and decrease of power should be mild.

A.4.3 Additional Rules for Sprinters

- Sprinters specialize in explosive Power, so when climbing short ranged hills, sprinters are suggested to take the opportunity and ride hard to create good results and then recover when altitude is decreasing.

A.5 Expected Result and Improvements

The model provide the expected time if certain cyclist follow the suggestions above, which is listed in the table

The result is just a rough prediction from the model, which serves as an reference for riders and Directeur Sporif. For further improvements, riders should strengthen daily practice to improve performance radically.

Gender	Type	Expected time/s
Male	Time Trail Specialist	8639.02
	Sprinter	10341.26
Female	Time Trail Specialist	8448.36
	Sprinter	11097.82

Table 3: Expected result of MCM time trail couese